88117201

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 1

Wednesday 2 November 2011 (afternoon)

2 hours

Candidate session number

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INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your session number on each answer sheet, and attach them to this examination paper and your cover sheet using the tag provided.
- At the end of the examination, indicate the number of sheets used in the appropriate box on your cover sheet.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

*Answer **all** the questions in the spaces provided. Working may be continued below the lines, if necessary.*

1. *[Maximum mark: 6]*

From a vertex of an equilateral triangle of side $2x$, a circular arc is drawn to divide the triangle into two regions, as shown in the diagram below.

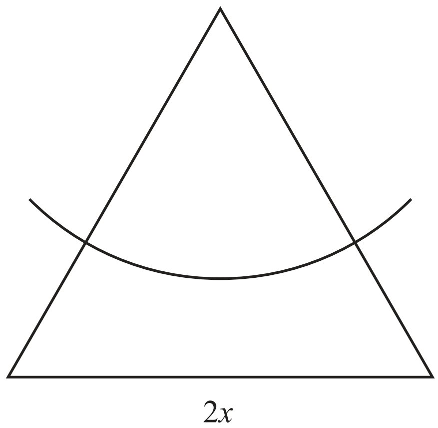

diagram not to scale

Given that the areas of the two regions are equal, find the radius of the arc in terms of x.

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2. *[Maximum mark: 6]*

Find the cube roots of i in the form $a+b\mathrm{i}$, where $a, b \in \mathbb{R}$.

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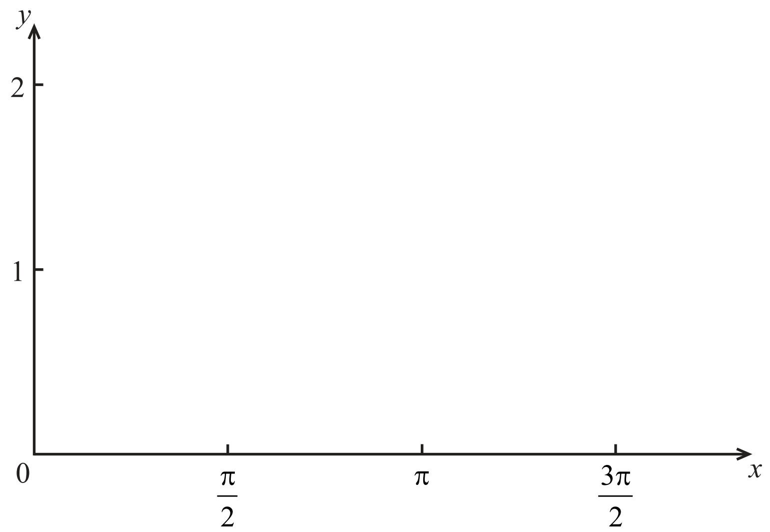
3. *[Maximum mark: 6]*

In a particular city 20 % of the inhabitants have been immunized against a certain disease. The probability of infection from the disease among those immunized is $\frac{1}{10}$, and among those not immunized the probability is $\frac{3}{4}$. If a person is chosen at random and found to be infected, find the probability that this person has been immunized.

4. *[Maximum mark: 6]*

Given that $f(x) = 1 + \sin x$, $0 \le x \le \frac{3\pi}{2}$,

(a) sketch the graph of f; *[1 mark]*



(b) show that $(f(x))^2 = \frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x$; *[1 mark]*

(c) find the volume of the solid formed when the graph of f is rotated through 2π radians about the x-axis. *[4 marks]*

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5. *[Maximum mark: 6]*

A target consists of three concentric circles of radii 1 m, 3 m and 5 m respectively, as shown in the diagram.

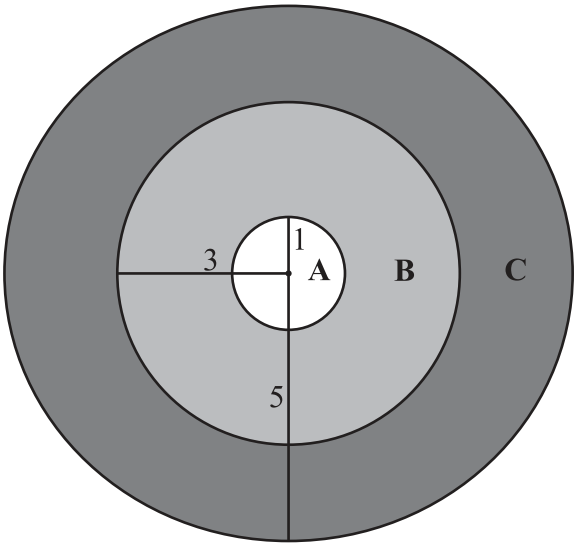

diagram not to scale

Nina shoots an arrow at the target. She has a probability of $\frac{1}{2}$ of hitting the target. If the arrow hits the target it does so at a random point on the target. Ten points are scored for hitting region A, six points for hitting region B, and three points for hitting region C. Find the expected number of points Nina scores each time she shoots an arrow at the target.

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6. *[Maximum mark: 7]*

Given that $y = \frac{1}{1-x}$, use mathematical induction to prove that $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = \frac{n!}{(1-x)^{n+1}}$, $n \in \mathbb{Z}^+$.

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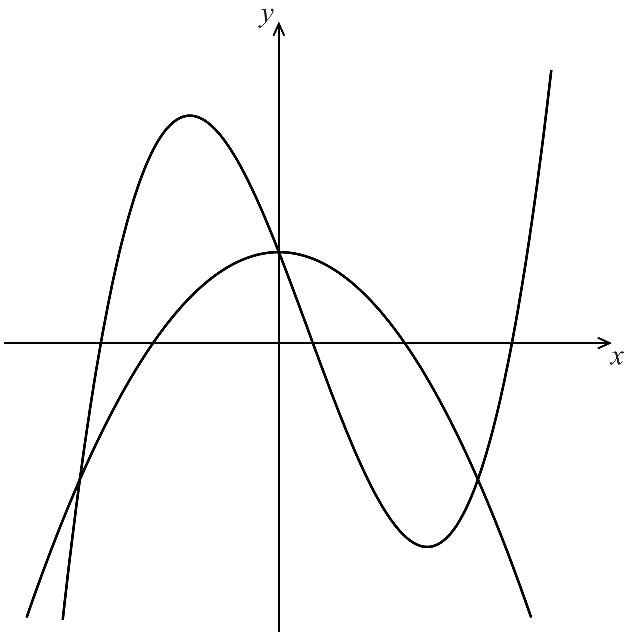
7. *[Maximum mark: 7]*

The graphs of $f(x) = -x^2 + 2$ and $g(x) = x^3 - x^2 - bx + 2$, $b > 0$, intersect and create two closed regions. Show that these two regions have equal areas.



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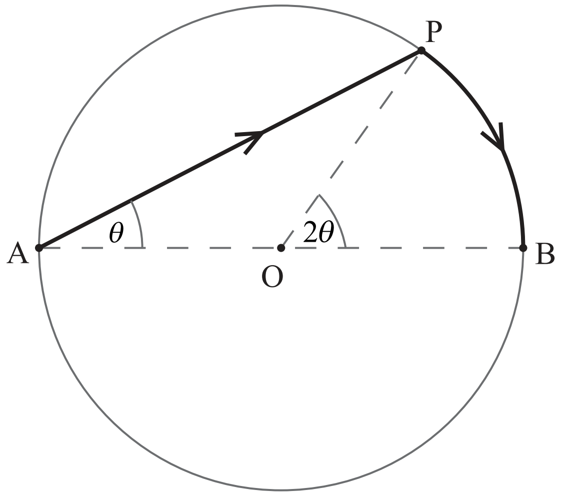
8. *[Maximum mark: 8]*

The diagram below shows a circular lake with centre O, diameter AB and radius 2 km.



Jorg needs to get from A to B as quickly as possible. He considers rowing to point P and then walking to point B. He can row at 3 km h^{-1} and walk at 6 km h^{-1}. Let $\text{P}\hat{\text{A}}\text{B} = \theta$ radians, and t be the time in hours taken by Jorg to travel from A to B.

(a) Show that $t = \frac{2}{3}(2\cos\theta + \theta)$. *[3 marks]*

(b) Find the value of θ for which $\frac{\mathrm{d}t}{\mathrm{d}\theta} = 0$. *[2 marks]*

(c) What route should Jorg take to travel from A to B in the least amount of time? Give reasons for your answer. *[3 marks]*

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9. *[Maximum mark: 8]*

Consider the equation $yx^2 + (y-1)x + (y-1) = 0$.

(a) Find the set of values of y for which this equation has real roots. *[4 marks]*

(b) Hence determine the range of the function $f : x \to \frac{x+1}{x^2+x+1}$. *[3 marks]*

(c) Explain why f has no inverse. *[1 mark]*

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*Do **NOT** write solutions on this page. Any working on this page will **NOT** be marked.*

SECTION B

*Answer **all** the questions on the answer sheets provided. Please start each question on a new page.*

10. *[Maximum mark: 10]*

A continuous random variable X has the probability density function

$$f(x) = \begin{cases} k \sin x, & 0 \le x \le \dfrac{\pi}{2} \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of k. *[2 marks]*

(b) Find $\mathrm{E}(X)$. *[5 marks]*

(c) Find the median of X. *[3 marks]*

11. *[Maximum mark: 11]*

At 12:00 a boat is 20 km due south of a freighter. The boat is travelling due east at 20 km h^{-1}, and the freighter is travelling due south at 40 km h^{-1}.

(a) Determine the time at which the two ships are closest to one another, and justify your answer. *[8 marks]*

(b) If the visibility at sea is 9 km, determine whether or not the captains of the two ships can ever see each other's ship. *[3 marks]*

*Do **NOT** write solutions on this page. Any working on this page will **NOT** be marked.*

12. *[Maximum mark: 15]*

(a) For non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, show that

(i) if $|\boldsymbol{a}-\boldsymbol{b}| = |\boldsymbol{a}+\boldsymbol{b}|$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular;

(ii) $|\boldsymbol{a}\times\boldsymbol{b}|^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2 - (\boldsymbol{a}\bullet\boldsymbol{b})^2$. *[8 marks]*

(b) The points A, B and C have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

(i) Show that the area of triangle ABC is $\frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{b}\times\boldsymbol{c}+\boldsymbol{c}\times\boldsymbol{a}|$.

(ii) Hence, show that the shortest distance from B to AC is

$$\frac{|\boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{b}\times\boldsymbol{c}+\boldsymbol{c}\times\boldsymbol{a}|}{|\boldsymbol{c}-\boldsymbol{a}|}.$$ *[7 marks]*

13. *[Maximum mark: 24]*

The curve C with equation $y = f(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{\ln y}(x+2),\ y > 1,$$

and $y = \mathrm{e}$ when $x = 2$.

(a) Find the equation of the tangent to C at the point $(2, \mathrm{e})$. *[3 marks]*

(b) Find $f(x)$. *[11 marks]*

(c) Determine the largest possible domain of f. *[6 marks]*

(d) Show that the equation $f(x) = f'(x)$ has no solution. *[4 marks]*